The Open University

M338
Diagnostic quiz

Am I ready to start M338, *Topology?*

Purpose of this document

This Diagnostic Quiz is designed to help you to answer the question of whether you are ready to start the course M338, *Topology.* The topics which are included in the quiz are those that we expect you to be reasonably familiar with before you start the course.

We suggest that you try this quiz first without looking at any books, and only look at a book when you are stuck. You will not necessarily remember everything, and may well need to look up some things. This is perfectly all right, as M338 contains a Handbook for you to use during your study of the course; you only need to check that you are able to use it.

This document also contains some advice on preparatory work that you may find useful before starting M338, *Topology.*

About the quiz

The questions are grouped into several sections, each on a specific topic. You do not need to try all the questions in a section if you are certain that you could tackle them successfully. Some questions can be answered in more than one way, so do not immediately think that your solution is incorrect simply because it is different from the one that we supply! If you can complete the quiz in a reasonable time, with only the occasional need to look at other material, then you should be well prepared for M338. If you found some areas that you remember having met before but need to do some more work on, then you should consider the suggestions for additional study that appear after the solutions to the quiz.

M338, *Topology* is a fascinating course, covering topics that are central to much modern pure mathematics. The better prepared you are for it the more time you will have to enjoy the mathematics, and the greater your chance of success. Try the quiz questions now, and then read the worked solutions that start on page 6 of this document to see if you are ready for M338.

If, after working through this quiz, you are not quite sure whether M338 is the right course for you, then we urge you to seek further help and advice either from your Regional Enquiry Service or from a Mathematics Staff Tutor at your local Regional Centre. The main OU 'Curriculum and Qualifications' website, www3.open.ac.uk/courses, gives you access to telephone numbers and email addresses for these.

M338, *Topology:* Am I ready? quiz

Proofs

1 Use proof by contradiction to prove that, if $n = a + 2b$, where a and b are positive real numbers, then either $a \geq \frac{1}{2}n$ or $b \geq \frac{1}{4}n$.

2 Write down (a) the converse and (b) the contrapositive of the following implication:

If $2^n - 1$ is prime, where n is a positive integer, then n is prime.

Use part (b) to prove the implication.

NOTE: By 'the contrapositive of the statement "$A \Rightarrow B$"' we mean the statement "not $B \Rightarrow$ not A".

Inequalities

3 Prove the following inequalities:

(a) $\sqrt{a^2 + b^2} \leq a + b$, for any non-negative real numbers a and b;

(b) $|a + b| \leq |a| + |b|$, for any real numbers a and b.

Sets

4 For each pair of sets A and B below, find $A \cup B$, $A \cap B$, $A - B$ and $B - A$.

(a) $A = \{0, 2, 4, 6\}$, $B = \{4, 5, 6, 7\}$

(b) $A = [-2, 3)$, $B = (1, 7]$

5 Sketch the following sets.

(a) $A = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\right\}$

(b) $B = \left\{(x, y) \in \mathbb{R}^2 : (x + 1)^2 + 4y^2 + 1 \leq 0\right\}$

(c) $C = \left\{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y < 1\right\}$

6 Determine whether either of the sets A and B in each pair below is a subset of the other.

(a) $A = \left\{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 1\right\}$,

$B = \left\{(x, y) \in \mathbb{R}^2 : x^2 + (y-1)^2 < 1\right\}$

(b) $A = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\right\}$,

$B = \left\{(x, y) \in \mathbb{R}^2 : \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \le \frac{1}{16}\right\}$

7 Which of the following are (i) open intervals, (ii) closed intervals, or (iii) neither open nor closed intervals?

(a) $[0, 2]$ (b) $(0, 2]$ (c) $(0, 2)$ (d) $\mathbb{R}$ (e) $\varnothing$

(f) $(0, 2) \cap [1, 3]$ (g) $(0, 2) \cup [1, 3]$ (h) $\mathbb{R} - [3, \infty)$

(i) $(0, 2) \cup (3, 7)$

8 Which of the following sets are (i) bounded? (ii) unbounded? (iii) neither bounded nor unbounded?

(a) $[0, 2]$ (b) $\mathbb{R}$ (c) $(0, 2) \cap [1, 3]$

(d) $\mathbb{R} - [3, \infty)$ (e) $A = \left\{(x, y) \in \mathbb{R}^2 : |xy| \le 1\right\}$

9 For each of the following sets E, determine $\max E$ and $\min E$.

(a) $E = [0, 2)$ (b) $E = \{\frac{1}{n} : n \in \mathbb{N}\}$

10 For each of the following sets E, determine $\sup E$ and $\inf E$.

(a) $E = [0, 2)$ (b) $E = \{\frac{1}{n} : n \in \mathbb{N}\}$

Sequences

11 For each of the following sequences $\{a_n\}$, determine whether it (i) converges, (ii) diverges, or (iii) neither converges nor diverges.

(a) $\left\{\dfrac{2n+1}{3n^2-4}\right\}$ (b) $\left\{\dfrac{n! - 99}{2^n - 74}\right\}$ (c) $\{2^{1/n}\}$

12 The following statements are either true or false. Prove each statement that is true. For each statement that is false, give a specific example to show that it is false.

(a) $a_n \to \ell \Rightarrow a_n^2 \to \ell^2$

(b) $a_n^2 \to \ell^2,\ \ell > 0 \Rightarrow a_n \to \ell$

Relations

An *equivalence relation* on a set X is a relation $\sim$ on X which satisfies the following three properties:

REFLEXIVE	For all $x, y \in X$, $x \sim x$;
SYMMETRIC	For all $x, y \in X$, if $x \sim y$ then $y \sim x$;
TRANSITIVE	For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$ then $x \sim z$.

The *equivalence class* of $x \in X$ is the set $\{y \in X : x \sim y\}$.

13 Determine whether the following relation $\sim$ on a set A is an equivalence relation. If it is, find its equivalence classes.

$A = \mathbb{Z}$, $x \sim y$ if $x - y$ is odd.

14 For the following functions f and g, determine $f \circ f$, $f \circ g$ and $g \circ f$.

$$f: \mathbb{R} \to \mathbb{R} \quad x \mapsto x^2 + x + 1 \qquad \text{and} \qquad g: \mathbb{R} \to \mathbb{R} \quad x \mapsto x^3 - x$$

NOTE: $(f \circ g)(x)$ means $f(g(x))$.

15 For each of the following functions f, determine whether f has an inverse function f^{-1}. If f^{-1} exists, find it.

(a) $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^3 - 3$

(b) $f: \mathbb{R}^2 \to \mathbb{R}^2$
$(x, y) \mapsto (x + 1, y - 2)$

(c) $f: \mathbb{R}^2 \to \mathbb{R}^2$
$(x, y) \mapsto (x^2, y^2)$

16 For each of the following functions f and sets X, determine the set $f^{-1}(X)$.

(a) $f: \mathbb{R} \to \mathbb{R}$, $X = [-4, 5)$
$x \mapsto x^3 - 3$

(b) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $X = \{(x, y): x^2 + y^2 \leq 4\}$
$(x, y) \mapsto (x + 1, y - 2)$

Hint: You may find it useful to refer back to parts of the previous two questions!

17 Write down the definition of the statement 'f is continuous at an interior point c of an interval I in $\mathbb{R}$'.

18 Some of the following statements are true and some are false. Prove each statement that is true. For each statement that is false, give a specific example to show that it is false. (Throughout you may assume that the functions f and g are defined on some neighbourhood of a point c in $\mathbb{R}$.)

(a) If f and g are continuous at c, then $f + g$ is continuous at c.

(b) If f and $f + g$ are continuous at c, then g is continuous at c.

(c) If f is continuous at c and g is discontinuous at c, then $f + g$ is continuous at c.

(d) If f is continuous at c and g is discontinuous at c, then fg is continuous at c.

(e) If f is discontinuous at c and g is discontinuous at c, then $f + g$ is continuous at c.

(f) If f is discontinuous at c and g is discontinuous at c, then fg is discontinuous at c.

19 Some of the following statements are true and some are false. Prove each statement that is true. For each statement that is false, give a specific example to show that it is false.

(a) If f is continuous on $[0, 2)$, then f has an inverse function on $f([0, 2))$.

(b) If f is continuous on $[0, 2)$, then f is bounded on $[0, 2)$.

(c) If f is continuous on $[0, 2]$, then f is bounded on $[0, 2]$.

(d) If f is bounded on $[0, 2]$, then f is continuous on $(0, 2)$.

(e) If f is bounded on $[0, 2]$, then f is continuous at 1.

Diagnostic Quiz – Answers

1 Suppose that $n = a + 2b$, where a and b are positive real numbers. Suppose also that $a < \frac{1}{2}n$ and $b < \frac{1}{4}n$. Then

$$\begin{aligned} n &= a + 2b \\ &< \tfrac{1}{2}n + 2\left(\tfrac{1}{4}n\right) = n. \end{aligned}$$

This contradiction shows that the supposition that $a < \frac{1}{2}n$ and $b < \frac{1}{4}n$ must be false. That is, we must have either $a \geq \frac{1}{2}n$ or $b \geq \frac{1}{4}n$.

2 (a) The converse of the implication is the following implication:

If n is prime, then $2^n - 1$ is prime.

(b) The contrapositive of the implication is the following implication:

If n is not prime, then $2^n - 1$ is not prime.

We now prove the contrapositive of the original implication.

Suppose that n is a positive integer that is not prime. If $n = 1$, then $2^n - 1 = 2 - 1 = 1$, which is not prime.

If $n > 1$, then $n = ab$ where $1 < a, b < n$. It follows that

$$\begin{aligned} 2^n - 1 &= 2^{ab} - 1 \\ &= (2^a)^b - 1 \\ &= (2^a - 1)\left((2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1\right), \end{aligned}$$

on factoring.

Now $2^a - 1 > 1$, since $a > 1$; and similarly $(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1 > 1$, since both a and b are greater than 1. Hence $2^n - 1$ is not prime.

We have thus proved the required contrapositive implication in both the cases $n = 1$ and $n > 1$. It follows that the original implication is also true, for any positive integer n.

3 For convenience, we shall use the symbol '$\Leftrightarrow$' to denote 'if and only if'.

(a) Here

$$\begin{aligned} & \sqrt{a^2 + b^2} \leq a + b \\ \Leftrightarrow\quad & a^2 + b^2 \leq (a + b)^2, \text{ for non-negative } a \text{ and } b \\ & \qquad\quad = a^2 + 2ab + b^2 \\ \Leftrightarrow\quad & 0 \leq 2ab, \text{ on rearranging.} \end{aligned}$$

Since a and b are both non-negative, it follows that $2ab \geq 0$. The above chain of equivalent statements shows that we must then also have $\sqrt{a^2 + b^2} \leq a + b$.

(b) Here

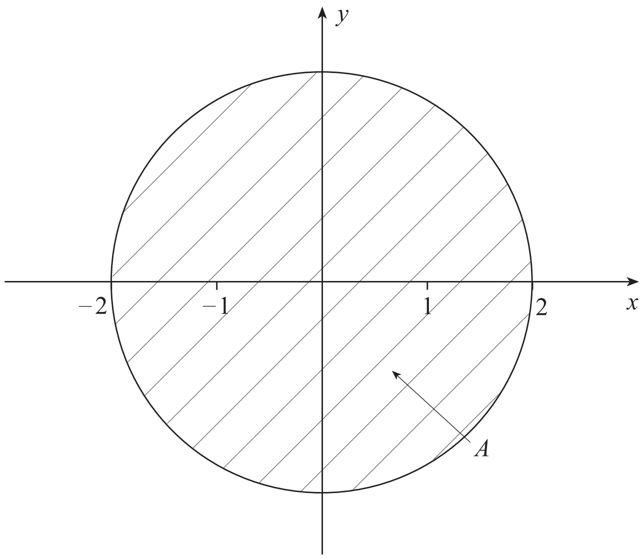
$$\begin{aligned} & |a + b| \leq |a| + |b| \\ \Leftrightarrow\ & |a + b|^2 \leq (|a| + |b|)^2 \\ \Leftrightarrow\ & (a + b)^2 \leq (|a| + |b|)^2 \\ \Leftrightarrow\ & a^2 + 2ab + b^2 \leq a^2 + 2|a| \times |b| + b^2 \\ \Leftrightarrow\ & 2ab \leq 2|a| \times |b|. \end{aligned}$$

For any real numbers a and b, we have that $2ab \leq 2|a| \times |b|$. The above chain of equivalent statements shows that we must then also have $|a + b| \leq |a| + |b|$.

4 (a) $A \cup B = \{0, 2, 4, 5, 6, 7\}$, $A \cap B = \{4, 6\}$, $A - B = \{0, 2\}$ and $B - A = \{5, 7\}$.

(b) $A \cup B = [-2, 7]$, $A \cap B = (1, 3)$, $A - B = [-2, 1]$ and $B - A = [3, 7]$.

5 (a) The set A is the interior of the circle with centre $(0, 0)$ and radius 2, together with the boundary of the circle:

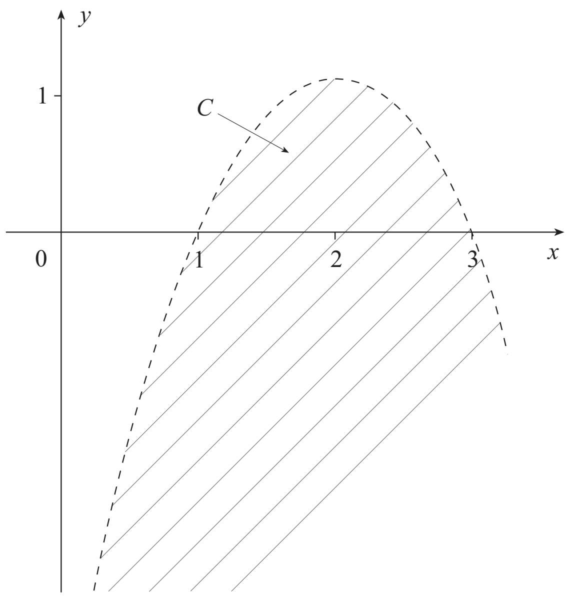


(b) For any real x and y, $(x + 1)^2$ and $4y^2$ are non-negative, so that

$$(x + 1)^2 + 4y^2 + 1 \geq 1.$$

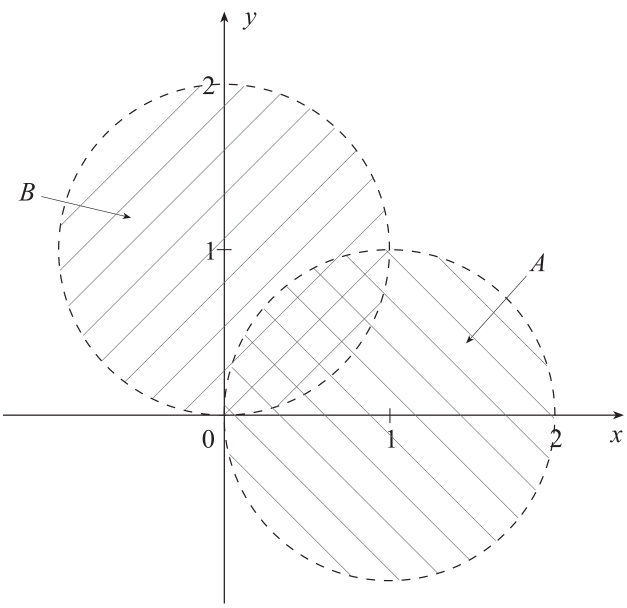
It follows that there are no points (x, y) in $\mathbb{R}^2$ for which $(x + 1)^2 + 4y^2 + 1 \leq 0$, so that $B = \varnothing$, the empty set.

(c) The set C is the 'inside' of a parabola, but does not include the parabola itself:



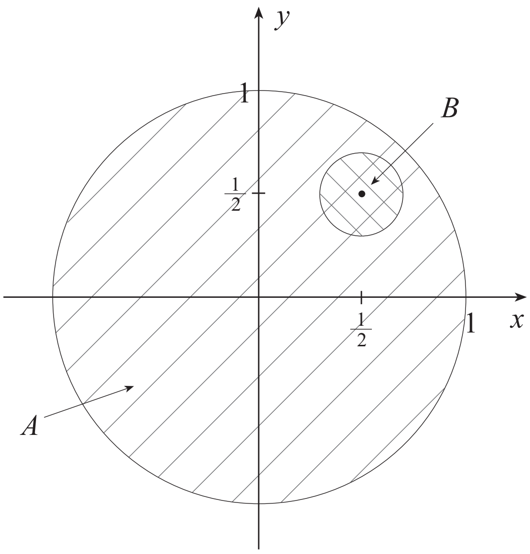
6 (a) A is the interior of a circle with centre $(1, 0)$ and radius 1, and B is the interior of a circle with centre $(0, 1)$ and radius 1.

The point $(0, \frac{1}{2})$ belongs to B but not to A, and the point $(\frac{1}{2}, 0)$ belongs to A but not to B. It follows that neither A nor B is a subset of the other.



(b) A is the interior and boundary of a circle with centre $(0, 0)$ and radius 1, and B is the interior and boundary of a circle with centre $(\frac{1}{2}, \frac{1}{2})$ and radius $\frac{1}{4}$.

It is clear from the diagram that B is a subset of A.



7 (a) $[0, 2]$ is a closed interval.

(b) $(0, 2]$ is an interval, but is neither open nor closed.

(c) $(0, 2)$ is an open interval.

(d) $\mathbb{R}$ is an interval. By convention, it is both open and closed.

(e) $\varnothing$ is the empty set. It is not an interval.

(f) $(0, 2) \cap [1, 3]$ is the interval $[1, 2)$, which is neither open nor closed.

(g) $(0, 2) \cup [1, 3]$ is the interval $(0, 3]$, which is neither open nor closed.

(h) $\mathbb{R} - [3, \infty)$ is the interval $(-\infty, 3)$; this is an open interval.

(i) $(0, 2) \cup (3, 7)$ is not an interval, and hence is not an open interval nor a closed interval.

8 Note that no set can be 'neither bounded nor unbounded'!

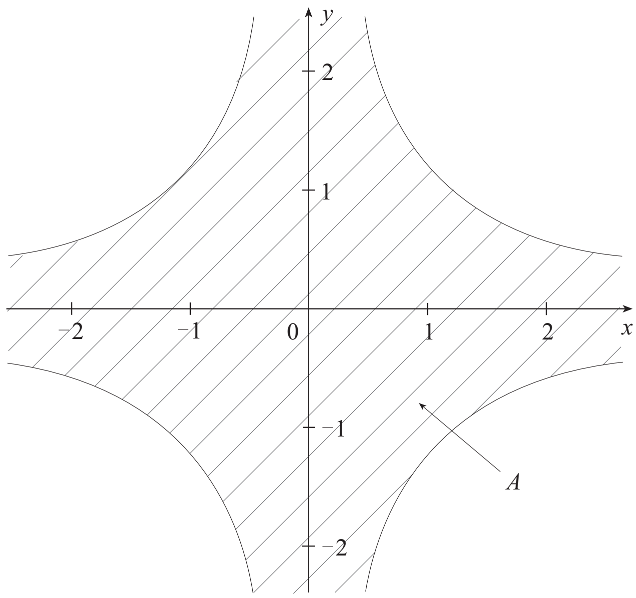
(a) $[0, 2]$ is a bounded set.

(b) $\mathbb{R}$ is an unbounded set.

(c) $(0, 2) \cap [1, 3] = [1, 2)$ is a bounded set.

(d) $\mathbb{R} - [3, \infty) = (-\infty, 3)$ is an unbounded set.

(e) $A = \{(x, y) \in \mathbb{R}^2 \colon |xy| \leq 1\}$ is the region containing the origin bounded by the rectangular hyperbola with equation $xy = 1$ and its reflection in the x-axis (viz. the curve with equation $xy = -1$, together with the hyperbolas themselves). It is an unbounded set.



9 (a) 2 has the property that $x \leq 2$ for all $x \in E = [0, 2)$, but no smaller number M is such that $x \leq M$ for all $x \in E$; however 2 does not belong to E. Therefore E has no maximum.

0 is the minimum of E, since 0 belongs to E and

$$x \geq 0 \quad \text{for all } x \in E.$$

(b) 1 is the maximum of $E = \{\frac{1}{n} \colon n \in \mathbb{N}\}$, since 1 belongs to E and

$$x \leq 1 \quad \text{for all } x \in E.$$

0 has the property that $x \geq 0$ for all $x \in E = \{\frac{1}{n} \colon n \in \mathbb{N}\}$, but no larger number m is such that $x \geq m$ for all $x \in E$; however 0 does not belong to E. Therefore E has no minimum.

10 (a) 2 has the property that $x \le 2$ for all $x \in E = [0, 2)$, and so is an upper bound of E. To show that 2 is the least upper bound (or supremum) of E, we must prove that any number $M < 2$ is not an upper bound of E. To do this, we must find an element x in $[0, 2)$ which is greater than M.

But, if $M < 2$, then there is a number $x \in E = [0, 2)$ such that

$$M < x \quad \text{and} \quad x < 2;$$

for example, the number $\max\{0, \frac{1}{2}(M + 2)\}$ will serve for this purpose. Hence M cannot be an upper bound of E. Therefore 2 is the least upper bound, or supremum, of E.

Since 0 is the minimum of E, it is also the greatest lower bound (or infimum) of E.

(b) Since 1 is the maximum of E, it is also the least upper bound (or supremum) of E.

0 has the property that $x \ge 0$ for all $x \in E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, and so is a lower bound of E. To show that 0 is the greatest lower bound (or infimum) of E, we must prove that any number $m > 0$ is not a lower bound of E. To do this, we must find an element $x = \frac{1}{n}$ in $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ which is less than m; that is, such that

$$\tfrac{1}{n} < m \quad \text{or} \quad n > \tfrac{1}{m}.$$

We can certainly choose n so that the inequality $n > \frac{1}{m}$ holds, by the Archimedean Property of the real numbers, and so we can choose n so that the inequality $\frac{1}{n} < m$ holds. Therefore 0 is the greatest lower bound, or infimum, of E.

11 In our solution we will use the fact that the following sequences are null sequences, that is, sequences that tend to 0 as $n \to \infty$:

$$\left\{\frac{1}{n^p}\right\}, \text{for } p > 0; \quad \left\{\frac{c^n}{n!}\right\}, \text{for any real } c.$$

(a) Let $a_n = \dfrac{2n+1}{3n^2-4}$, $n \ge 1$. The 'dominant' term here is the n^2 on the denominator, so we divide both numerator and denominator by n^2, so that

$$\begin{aligned} a_n = \frac{2n+1}{3n^2-4} &= \frac{2 \times \left(\frac{1}{n}\right) + \frac{1}{n^2}}{3 - 4 \times \left(\frac{1}{n^2}\right)} \\ &\to \frac{2 \times 0 + 0}{3 - 4 \times 0}, \quad \text{as } n \to \infty \\ &= \frac{0}{3} = 0. \end{aligned}$$

Hence the sequence $\{a_n\}$ converges to 0.

(b) Let $a_n = \dfrac{n! - 99}{2^n - 74}$, $n \ge 1$. The 'dominant' term here is the $n!$ on the numerator, so we look at the reciprocal sequence $\{1/a_n\}$ and divide both numerator and denominator by $n!$. Thus

$$\begin{aligned} \frac{1}{a_n} = \frac{2^n - 74}{n! - 99} &= \frac{\frac{2^n}{n!} - 74 \times \frac{1}{n!}}{1 - 99 \times \frac{1}{n!}} \\ &\to \frac{0 - 74 \times 0}{1 - 99 \times 0}, \quad \text{as } n \to \infty \\ &= \frac{0}{1} = 0. \end{aligned}$$

Hence the sequence $\left\{\frac{1}{a_n}\right\}$ converges to 0. It follows that, since $a_n > 0$ for all $n \ge 1$, the original sequence $\{a_n\}$ tends to ∞ as $n \to \infty$. In particular, $\{a_n\}$ is divergent.

(c) As $n \to \infty$, $\dfrac{1}{n} \to 0$ so that

$$\begin{aligned} 2^{\frac{1}{n}} &\to 2^0 \\ &= 1. \end{aligned}$$

(Here we have used the fact that the function $f(x) = 2^x$ is continuous at 0.) In particular, the sequence $\{a_n\}$ is convergent (to 1).

12 (a) This statement is true.

It is an immediate consequence of the Product Rule for convergent sequences. (Recall that a sequence $\{a_n\}$ is convergent (with limit ℓ) if, for each positive number ε, there is a number X such that $|a_n - \ell| < \varepsilon$, for all $n > X$.)

Alternatively, we can prove it directly, as follows. Since the sequence $\{a_n\}$ is convergent, then:

(1) there is some positive number K such that $|a_n| < K$ for all $n \ge 1$, and

(2) for any positive number ε, there is a positive integer N such that

$$|a_n - \ell| < \frac{\varepsilon}{K + |\ell|} \quad \text{for all } n > N.$$

So, let ε be any positive number and N a positive integer chosen so that (b) above holds. It then follows from (a) and (b) above that, for $n > N$, we have

$$\begin{aligned} \left|a_n^2 - \ell^2\right| &= |(a_n - \ell) \times (a_n + \ell)| \\ &= |a_n - \ell| \times |a_n + \ell| \\ &< \frac{\varepsilon}{K + |\ell|} \times (|a_n| + |\ell|) \\ &< \frac{\varepsilon}{K + |\ell|} \times (K + |\ell|) = \varepsilon. \end{aligned}$$

This completes the proof that the sequence $\{a_n^2\}$ is convergent to ℓ^2.

(b) This statement is false.

For example, let $a_n = (-1)^n$, for $n \ge 1$, and $\ell = 1$. Then $(a_n)^2 = 1$ for $n \ge 1$, so that $a_n^2 \to 1 = \ell^2$ as $n \to \infty$. However the subsequence $a_{2k+1} = (-1)^{2k+1} = -1$, for $k \ge 1$, converges to the limit -1, which is different from 1. It follows that, although $a_n^2 \to 1^2$ as $n \to \infty$, nevertheless $a_n \not\to 1$ (indeed $\{a_n\}$ is not even convergent).

13 This relation is NOT reflexive. For example, $2 \not\sim 2$ since $2 - 2 = 0$, which is not odd.

The relation is symmetric. For, if $x \sim y$, then $x - y$ is odd, so that $y - x = -(x - y)$ is also odd and therefore $y \sim x$.

The relation is NOT transitive. For example, $5 \sim 2$ since $5 - 2 = 3$, which is odd, and $2 \sim 1$ since $2 - 1 = 1$, which is odd – however $5 \not\sim 1$ since $5 - 1 = 4$, which is not odd.

Thus the relation is NOT an equivalence relation.

You only need to show that one of the conditions fails, to conclude that $\sim$ is not an equivalence relation — you don't need to check them all!

14 All the functions $f \circ f$, $f \circ g$ and $g \circ f$ have domain $\mathbb{R}$ and codomain $\mathbb{R}$. Then:

$$\begin{aligned} f \circ f : x \mapsto\ & f(f(x)) \\ &= f(x)^2 + f(x) + 1 \\ &= \left(x^2 + x + 1\right)^2 + \left(x^2 + x + 1\right) + 1 \\ &= \left(x^4 + 2x^3 + 3x^2 + 2x + 1\right) + \left(x^2 + x + 1\right) + 1 \\ &= x^4 + 2x^3 + 4x^2 + 3x + 3; \end{aligned}$$

$$\begin{aligned} f \circ g : x \mapsto\ & f(g(x)) \\ &= g(x)^2 + g(x) + 1 \\ &= \left(x^3 - x\right)^2 + \left(x^3 - x\right) + 1 \\ &= \left(x^6 - 2x^4 + x^2\right) + \left(x^3 - x\right) + 1 \\ &= x^6 - 2x^4 + x^3 + x^2 - x + 1; \text{ and} \end{aligned}$$

$$\begin{aligned} g \circ f : x \mapsto\ & g(f(x)) \\ &= f(x)^3 - f(x) \\ &= \left(x^2 + x + 1\right)^3 - \left(x^2 + x + 1\right) \\ &= \left(x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1\right) \\ &\quad - \left(x^2 + x + 1\right) \\ &= x^6 + 3x^5 + 6x^4 + 7x^3 + 5x^2 + 2x. \end{aligned}$$

15 (a) The function f is one-one. For, if $f(x_1) = f(x_2)$, then

$$x_1^3 - 3 = x_2^3 - 3 \text{ so that } x_1^3 = x_2^3, \text{ and so } x_1 = x_2.$$

It follows that f has an inverse whose domain is $f(\mathbb{R})$. Then, either by sketching the graph of f or using the facts that $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, we see that $f(\mathbb{R}) = \mathbb{R}$.

Next, if $y = f(x) = x^3 - 3$, we have $x^3 = y + 3$ and so $x = \sqrt[3]{y + 3}$. It follows that f^{-1} is the function:

$$\begin{aligned} f^{-1} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \sqrt[3]{x + 3} \end{aligned}$$

(b) The function f is one-one. For, if $f(x_1, y_1) = f(x_2, y_2)$, then

$$(x_1 + 1, y_1 - 2) = (x_2 + 1, y_2 - 2)$$

so that

$$(x_1, y_1) + (1, -2) = (x_2, y_2) + (1, -2),$$

and so $x_1 = x_2$ and $y_1 = y_2$.

It follows that f has an inverse whose domain is $f(\mathbb{R}^2)$. But $f(x, y)$ is simply (x, y) translated 1 to the right and 2 down, so that clearly $f(\mathbb{R}^2)$ is just $\mathbb{R}^2$ itself.

Next, if $f(x, y) = (x + 1, y - 2) = (X, Y)$, we have $f^{-1}(X, Y) = (x, y) = (X - 1, Y + 2)$. It follows that f^{-1} is the function:

$$\begin{aligned} f^{-1} : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (x - 1, y + 2) \end{aligned}$$

(c) The function f is not one-one, since $f(2, 3) = (4, 9)$ and $f(-2, -3) = (4, 9)$, although $(2, 3) \neq (-2, -3)$. It follows that f has no inverse function.

16 (a) From 15(a), the function f is one-one on $\mathbb{R}$. It is also strictly increasing on $\mathbb{R}$. Then, since

$$f^{-1}(-4) = \sqrt[3]{(-4) + 3} = -1 \text{ and}$$
$$f^{-1}(5) = \sqrt[3]{(5) + 3} = 2,$$

it follows that $f^{-1}([-4, 5)) = [-1, 2)$.

(b) You saw in Problem 15(b) that f^{-1} translates any point (x, y) distances 1 to the left and 2 up. Here X is the circle with centre $(0, 0)$ and radius 2, together with its inside. It follows that $f^{-1}(X)$ is also a circle of radius 2 (together with its inside), and that its centre is $f^{-1}((0, 0)) = (-1, 2)$. Thus $f^{-1}(X) = \left\{(x, y) : (x + 1)^2 + (y - 2)^2 \leq 4\right\}$.

17 f is continuous at an interior point c of an interval I in $\mathbb{R}$ if:

for each positive number ε, there is a positive number δ such that

$$|f(x) - f(c)| < \varepsilon \text{ for all } x \text{ in } I \text{ satisfying } |x - c| < \delta.$$

18 (a) This statement is true, and may be proved as follows.

The functions f, g and $f + g$ are certainly defined on some neighbourhood I of c. We want to prove that

for each positive number ε, there is a positive number δ such that

$$|(f(x) + g(x)) - (f(c) + g(c))| < \varepsilon$$
$$\text{for all } x \text{ satisfying } |x - c| < \delta. \qquad (1)$$

We know that, since f is continuous at c, there is a positive number δ_1 such that

$$|f(x) - f(c)| < \tfrac{1}{2}\varepsilon,$$
$$\text{for all } x \text{ in } I \text{ satisfying } |x - c| < \delta_1; \qquad (2)$$

similarly, since g is continuous at c, there is a positive number δ_2 such that

$$|g(x) - g(c)| < \tfrac{1}{2}\varepsilon,$$
$$\text{for all } x \text{ in } I \text{ satisfying } |x - c| < \delta_2. \qquad (3)$$

We now choose $\delta = \min\{\delta_1, \delta_2\}$. Then both statements (2) and (3) hold for all x in I satisfying $|x - c| < \delta$, so that

$$\begin{aligned} &|(f(x) + g(x)) - (f(c) + g(c))| \\ &\quad = |(f(x) - f(c)) + (g(x) - g(c))| \\ &\quad \leq |f(x) - f(c)| + |g(x) - g(c)| \\ &\quad < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon, \end{aligned}$$

so that the statement (1) holds.

Hence $f + g$ is continuous at c.

(b) This statement is true, and may be proved as follows.

Since f is continuous at c, so is $-f$ (this is easily proved from the definition of continuity, so we omit the details here).

Then, since $f+g$ and $-f$ are both continuous at c, it follows from part (a) that so also is the sum function $(f+g)+(-f)=g$.

(c) This statement is false.

For example, let $c=0$ and let f and g have domains $\mathbb{R}$ and be defined by the mappings

$$f\colon x \mapsto 0 \quad \text{and} \quad g\colon x \mapsto \begin{cases} -1, & \text{if } x \le 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Then in any neighbourhood of the point 0 we have $(f+g)(x)=g(x)$.

Then, since g is discontinuous at 0, so is $f+g$.

(d) This statement is false.

For example, let $c=0$ and let f and g have domains $\mathbb{R}$ and be defined by the mappings

$$f\colon x \mapsto 1 \quad \text{and} \quad g\colon x \mapsto \begin{cases} -1, & \text{if } x \le 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Then in any neighbourhood of the point 0 we have $(fg)(x)=g(x)$.

Then, since g is discontinuous at 0, so is fg.

(e) This statement is false.

For example, let $c=0$ and let f and g have domains $\mathbb{R}$ and be defined by the mappings

$$f\colon x \mapsto \begin{cases} -1, & \text{if } x \le 0, \\ 1, & \text{if } x > 0, \end{cases}$$

$$g\colon x \mapsto \begin{cases} -1, & \text{if } x \le 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Then

$$f+g\colon x \mapsto \begin{cases} -2, & \text{if } x \le 0, \\ 2, & \text{if } x > 0. \end{cases}$$

Then, both f and g are discontinuous at 0, and so is $f+g$.

(f) This statement is false.

For example, let $c=0$ and let f and g have domains $\mathbb{R}$ and be defined by the mappings

$$f\colon x \mapsto \begin{cases} -1, & \text{if } x \le 0, \\ 1, & \text{if } x > 0, \end{cases}$$

$$g\colon x \mapsto \begin{cases} -1, & \text{if } x \le 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Then $fg\colon x \mapsto 1$, for all $x \in \mathbb{R}$. This constant function fg is continuous at 0.

19 (a) This statement is false.

For example, the function $f(x)=\left(x-\frac{1}{2}\right)\left(x-\frac{3}{2}\right)$ on $[0,2)$ is not even one-one on $[0,2)$, since $f\left(\frac{1}{2}\right)=0=f\left(\frac{3}{2}\right)$; hence it has no inverse function on $[0,2)$.

(b) This statement is false.

For example, let

$$f\colon x \mapsto \frac{1}{x-2}, \quad x \in [0,2).$$

This function is continuous on $[0,2)$, but

$$f(x) \to -\infty \quad \text{as} \quad x \to 2^-.$$

(c) This statement is true, by the Boundedness Theorem for continuous functions.

(d) This statement is false.

For example, let

$$f\colon x \mapsto \begin{cases} -1, & \text{if } 0 \le x < 1, \\ 1, & \text{if } 1 \le x \le 2. \end{cases}$$

Then f is bounded (by 1) on $[0,2]$, but is discontinuous at the point 1; hence it is not continuous on $[0,2]$.

(e) This statement is false. A suitable example is the function f in part (d).

How did you get on?

Now that you have finished the quiz, how did you get on? The information below should help you to decide what, if anything, you should do next.

Most of this quiz was unfamiliar to me. I don't think I have met any of the topics before.	⇒	Perhaps you should study another course such as MU120, *Open Mathematics*, MST121, *Using Mathematics*, MS221, *Exploring Mathematics*, or M208, *Pure Mathematics*. You should contact a Regional Adviser to discuss this.
Some of these topics were familiar to me, but not all of them.	⇒	Perhaps you should study another course such as MST121, *Using Mathematics*, MS221, *Exploring Mathematics*, or M208, *Pure Mathematics*. You should contact a Regional Adviser to discuss this.
I have met all these topics before, but I am a bit rusty on some, and had to look up how to do a few of them.	⇒	You will benefit from doing some revision before starting M338. See below for some possible sources of help.
I could do all of these questions.	⇒	You are well prepared for M338. You can study it with confidence.

What resources are there to help me to revise/prepare for M338?

If you have studied the Open University course M208, *Pure Mathematics* before, then you could use parts of this to revise for M338, *Topology*.

If you need to brush up some of the topics in the quiz, then good places to do this are:

- For sets and mathematical proofs, read through Unit I2, *Mathematical Language*, of M208, *Pure Mathematics*
- For the other topics, dip into the book *A First Course in Mathematical Analysis*, by David Brannan. This was published by Cambridge University Press in 2006, and its ISBN number is 0-521-68424-2. (Equivalently you could read through Analysis Block A and Analysis Block B of M208, *Pure Mathematics*.)

You can also look for other resources on the web. The following useful websites were available when the Course Team checked; but we can offer no promise that they will continue to be available throughout the coming year! If you experience any problem with them, or if you come across other internet websites that offer appropriate assistance/information, please let us know via the email address mcs-course-enquiries@open.ac.uk, writing M338 in the subject line.

IRA, "Interactive Real Analysis"	http://web01.shu.edu/projects/reals/reals.html
Plymouth University	various URLs
Calculus on the Web, "COW"	http://cow.math.temple.edu/

NOTE: In the list below, the symbol "$\rightarrow A$" is used to mean that at this point you should click on the button "A".

Topic	Website URL and subsequent choices that you should make
Proofs and proof strategies; if and only if; converse; contrapositive	• http://zimmer.csufresno.edu/∼larryc/proofs/proofs.html
Proof by induction	• http://zimmer.csufresno.edu/∼larryc/proofs/proofs.mathinduction.html • http://www.tech.plym.ac.uk/maths/resources/PDFLaTeX/induction.pdf
Sets	• http://web01.shu.edu/projects/reals/reals.html → section 1.1
Maximum and minimum, supremum and infimum	• http://web01.shu.edu/projects/reals/reals.html → section 2.4
Sequences (of numbers)	• http://web01.shu.edu/projects/reals/reals.html → section 3.1 • http://cow.math.temple.edu/ → Calculus Book III → 1.Sequences and series → 1.Sequences → 1.Limits of sequences
Inequalities	• http://cow.math.temple.edu/ → Calculus Book I → 1.Functions and Geometry → 3.Inequalities • http://cow.math.temple.edu/ → Precalculus Book → 2.Equations → 2.Inequalities • http://www.tech.plym.ac.uk/maths/resources/PDFLaTeX/inequalities.pdf
Equivalence relations and classes	• http://web01.shu.edu/projects/reals/reals.html → section 1.3
Functions	• http://web01.shu.edu/projects/reals/reals.html → section 1.2 NOTE: This website uses the term 'range' where we use 'codomain'.
Composing functions	• http://cow.math.temple.edu/ → 1.Calculus Book I → 1.Functions and Geometry → 4.Functions → 1.Composition of Functions • http://cow.math.temple.edu/ → Precalculus Book → 5.Functions → 6.Function Operations → 1.Composition

Topic	Website URL and subsequent choices that you should make
Inverse functions	• http://cow.math.temple.edu/ → Precalculus Book → 5.Functions → 6.Function Operations → 2.Inverse functions
Limits of functions	• http://cow.math.temple.edu/ → Calculus Book 1 → 2.Limits and Continuity → 1.Ordinary Limits
Continuity	• http://web01.shu.edu/projects/reals/reals.html → sections 6.2 and 6.3
Properties of continuous functions	• http://web01.shu.edu/projects/reals/reals.html → section 6.2 and 6.4
Use of epsilon and delta	• http://cow.math.temple.edu/ → 1.Calculus Book I → 2.Limits and Continuity → 2.Continuity → 1.Epsilon and delta