The Open University

MT365
Diagnostic quiz

Am I ready to start MT365?

The questions that follow are designed to help you answer this.

Being an inter-disciplinary course, MT365 aims to be accessible to students coming from two different broad directions - mathematics and technology. It is therefore not as demanding in either of these directions as would be the case for a Level 3 course aimed specifically at mathematics or at technology students.

The course shows you how to use relatively simple mathematical ideas and processes to model a variety of naturally occurring problems, and obtain solutions that are sometimes the best possible, but are at any rate better than you could obtain without the methods described. The most important thing that you need in order to tackle this course is a willingness to get involved both with the mathematical ideas and with their application.

The questions are divided into two sections. The first is quite short, and deals with basic mathematical skills and techniques that you should have met in your previous studies. The second is a set of problems that appear in the first unit of the course, and give a flavour of the types of practical situations with which the course involves you.

If you find Part 1 difficult, then you may wish to consider taking an Open University mathematics course such as MST121 before proceeding to MT365.

As far as Part 2 is concerned, do not be put off if you cannot answer all of these problems; the question is, did you find them interesting? If so, and if you are happy with the ideas of Part 1, then you are likely to enjoy studying MT365.

WEB S00250 1
1.1

Diagnostic Quiz Questions

Part 1

1.1 Find:

(a) $2^6 \times 2^{-3}$; (b) $2^n \div 2^3$; (c) $k^4 \times k^7 \times (k^2)^{-1}$.

1.2 Simplify the following inequalities:

(a) $2(x+2) \leq 5x+1$;

(b) $\dfrac{1}{x} - \dfrac{1}{4} < 0$ (under the assumption that x is positive).

1.3 Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 2 & 1 & 7 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 3 & 1 \\ 2 & 1 & 1 \\ 3 & -3 & 1 \end{bmatrix}.$$

(a) Calculate the matrix product $\mathbf{AB}$.

(b) Why does the matrix product $\mathbf{BA}$ not make sense?

1.4 Matrix arithmetic can be performed modulo 2; that is, using just the numbers 0 and 1 as entries, where calculations are carried out as follows:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

Let $\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$, $\mathbf{C}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{C}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Find the products $\mathbf{CC}_1$ and $\mathbf{CC}_2$, with matrix arithmetic performed modulo 2.

1.5 The following are the coordinates of four of the corners of the unit cube:

$$(0,0,0), (0,0,1), (0,1,1) \text{ and } (1,1,0).$$

(a) What are the coordinates of the remaining corners?

(b) How many edges does the cube have?

1.6 Given that $b = v = 7$ and $k = 3$, and that the equations $bk = vr$ and $\lambda(v-1) = r(k-1)$ hold, find the value of λ.

1.7 A recurrence relation has

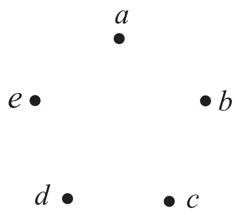
$$u_1 = 1, u_2 = 2 \text{ and}$$

$$u_n = 2u_{n-1} + (u_1 u_{n-2} + u_2 u_{n-3} + \cdots u_{n-3} u_2 + u_{n-2} u_1).$$

So $u_3 = 2u_2 + (u_1 u_1)$ and $u_4 = 2u_3 + (u_1 u_2 + u_2 u_1)$.

Find the values of u_5 and u_6.

1.8 Consider the following five dots.



How many different triangles are there with each corner at one of the dots? What is the answer for n dots? Explain briefly.

Part 2

2.1 Map Colouring

Consider the following map of the USA (excluding Alaska and Hawaii):

It is common for maps of this kind to be coloured in such a way that states (or countries) that share a common boundary line are coloured differently. This enables us to distinguish easily between the various states, and to locate the state boundaries. The question arises:

How many colours are needed to colour the entire map?

One might reasonably expect that the larger and more complicated a map, the more colours we might need to colour it (though, actually, this turns out not to be true).

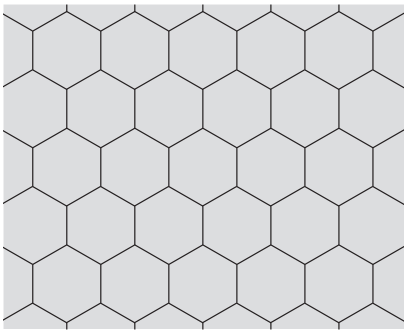
Can the above map of the USA be coloured with just three colours?

Hint Consider Nevada (shaded) and its neighbouring states.

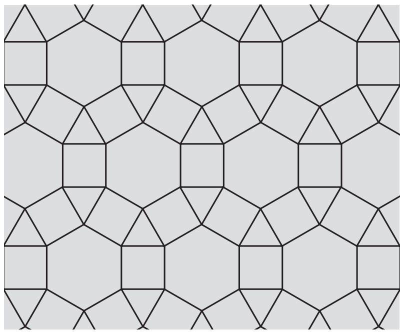
2.2 Tilings

If we attempt to tile a flat surface with tiles, we find that only certain shapes and arrangements are possible. Given a supply of tiles of assorted sizes and shapes, we cannot guarantee that they will all fit together neatly without gaps or overlaps. However, if all the tiles are regular polygons of the same shape and size, then we can determine whether such a tiling is possible.

Tiling (a) on the next page is a tiling with regular hexagons. Note that the tiles fit together without gaps or overlaps, and that the sides of neighbouring tiles match up exactly. Also, the arrangement of hexagons around each corner is the same. This tiling can be extended as far as we wish in all directions. Such a tiling by regular polygons is called a **regular tiling**.



(a)



(b)

We can also construct tilings from regular polygons of two or more different types. For example, tiling (b) above is constructed from equilateral triangles, squares and regular hexagons. Again, the arrangement of polygons around each corner is the same: hexagon, square, triangle, square. Such a tiling is called a **semi-regular tiling**.

(a) Construct a portion of the regular tiling consisting equilateral triangles.

(b) Construct a portion of a semi-regular tiling consisting entirely of squares and regular octagons (eight-sided polygons).

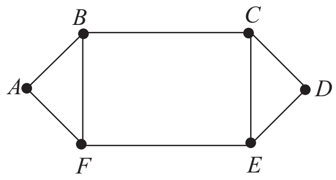
(c) Explain why it is *not* possible to construct a regular tiling using only regular pentagons (five-sided polygons).

2.3 Connection problems

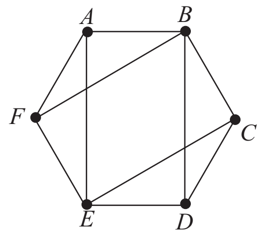
Much of MT365 is concerned with structures called *graphs*. These are *not* the plots of y against x with which you may well be familiar; graphs in MT365 are structures consisting of dots, some pairs of which are joined by lines. These graphs may represent a wide variety of practical situations in which we have a certain number of objects some pairs of which are linked in some way.

For example, a graph may represent a network of telephone exchanges, some pairs of which are *linked* directly by a cable. Hopefully, any exchange is linked to any other either directly or *via* one or more intermediate exchanges.

For example, here are two graphs that could represent simple telephone exchange networks.



(1)



(2)

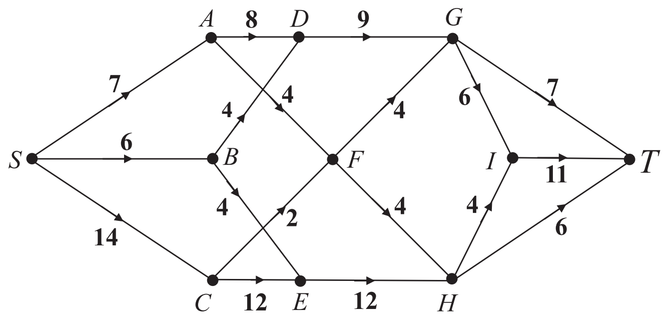
In each case, find:

(a) the smallest number of *links* whose closure would separate the network into parts that could not communicate with each other;

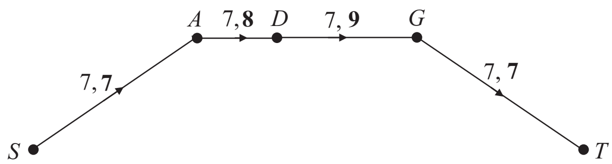
(b) the smallest number of *exchanges* whose closure would separate the *remaining* exchanges into two parts that could not communicate with each other.

2.4 Network flows

The following diagram represents a network of pipelines along which a fluid (for example, gas, oil or water) flows from a starting point S to a terminal T. Each of the intermediate points $A - I$ represents a pipe function at which the total flow into the junction must equal the total flow out (so that no fluid is 'lost' on the way). Each line between two junctions represents a pipeline, and the number next to it is the *capacity* of that pipeline (in some units of volume per unit time); the flow along a pipeline must not exceed its capacity, and must be in the direction indicated.



Inspection of the above diagram shows that a *flow* of at most 7 units can be sent along the route $SADGT$ without exceeding the capacity of any of the pipelines SA, AD, DG or GT. This is illustrated in the following diagram, where the first number on each line represents the flow along that pipeline and the second number – in bold type – its capacity.



(a) How can 13 units of fluid per unit time be sent from S to T without exceeding the capacity of any pipeline?

(b) How can 15 units per unit time be sent?

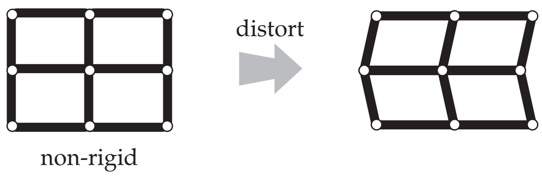
(c) Explain why it is impossible to send more than 23 units per unit time from S to T.

[*Hint* for (c): look at the pipelines DG, FG, HI and HT.]

2.5 Braced rectangular frameworks

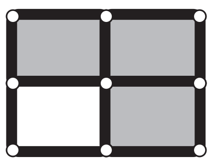
Many buildings are supported by rectangular steel frameworks, and it is important that such frameworks should remain rigid under heavy loads. One way to achieve this is to add *braces*, to prevent distortion.

For example, the following diagram shows how a simple unbraced rectangular framework can be distorted.

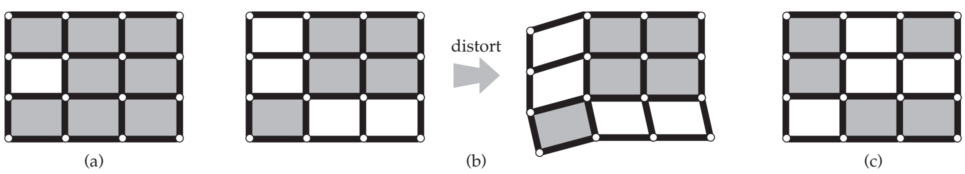


Now, adding only two braces, in the form of rectangular plates (indicated by shading) cannot make this framework rigid, as the following diagrams illustrate.

The minimum number of braces that we must add to make this framework rigid is three.



Now consider the following three frameworks:



(a) Framework (a) is rigid, but is over-braced, since some braces can be removed while maintaining its rigidity. Which brace(s) can be removed whilst maintaining rigidity?

(b) Framework (b) is not rigid, since it can be distorted as shown. Which position(s) have the property that a further brace in that position is sufficient to achieve rigidity?

(c) Is framework (c) rigid? If so, can any braces be removed while maintaining its rigidity? If not, how can it be made rigid by the addition of one further brace?

2.6 Job assignment

A building contractor advertises five jobs — those of bricklayer, carpenter, decorator, electrician and plumber. There are four applicants — one for carpenter and decorator, one for bricklayer, carpenter and plumber, one for decorator, electrician and plumber, and one for carpenter and electrician. Since each job is a full-time post, this means that not all the jobs can be filled. But is it possible for all four applicants to be assigned each to one job for which he or she is qualified?

In order to solve this problem, it is convenient to represent the information in tabular form, as shown below.

applicant	job
1	c, d
2	b, c, p
3	d, e, p
4	c, e

From the table, we can see that one possible assignment of applicants to jobs is:

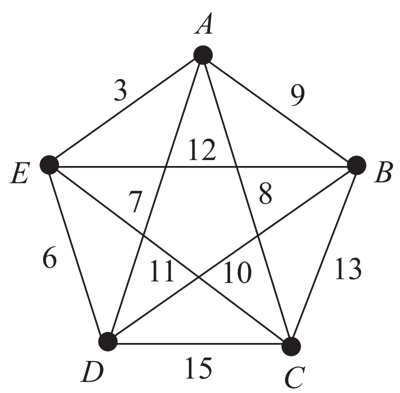
1 carpenter
2 plumber
3 decorator
4 electrician

(a) This is not the only solution; list as many other solutions as you can.

(b) Suppose that applicant 2 decides not to apply for the position of plumber. Is it still possible to assign the four applicants to jobs for which they have applied? If so, how can this be done? If not, how many positions can be filled?

2.7 Minimum connector problems

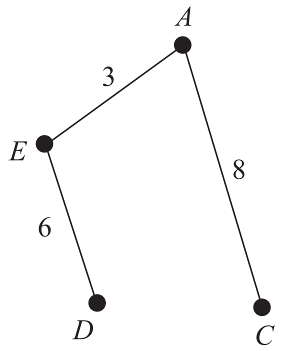
Consider the case of an electricity company that wants to lay a network of cables in order to link together five towns, A, B, C, D and E. It wants to minimize the amount of cabling, in order to keep its costs down. The distances (in miles) between the towns are shown in the following diagram:



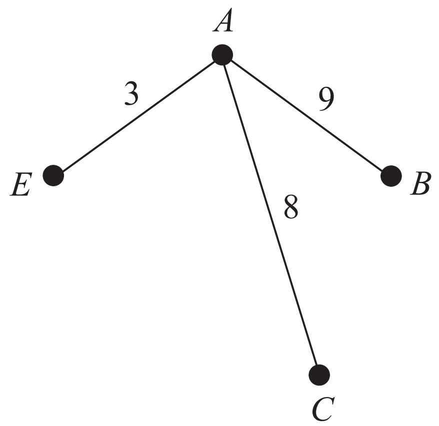
(For example, the distance between A and B is 9 miles and the distance between A and C is 8 miles.)

The company's problem is one of finding a *minimum connector* — a set of links of minimum *total* length that connect all five towns.

For example, a minimum connector that links the towns A, C, D and E (but not B) comprises the links AC, AE and DE. This minimum connector has total length 17 miles.



Similarly, a minimum connector that links the towns A, B, C and E (but not D) comprises the links AB, AC and AE, of total length 20 miles.



(a) Find a minimum connector that links the towns B, C, D and E.

(b) Find a minimum connector that links all five towns.

2.8 Travelling salesman problems

A travelling salesman wishes to visit a number of towns and return to his starting point, selling his wares as he goes. He wants to select the route with the least total length. Which route should he choose, and how long is it?

Although this type of problem sounds very like a minimum connector problem, it is actually much more difficult to calculate the best possible solution efficiently if there are a large number of cities.

However, solving the minimum connector problem for the same set of cities, *with one removed*, can give a *lower bound* for the solution to the problem. The final problem for you to try illustrates this rather subtle idea.

Look at the distances shown between towns A, B, C, D and E for the minimum connector problem above. Try to explain why the fact that the minimum connector for towns A, C, D and E has total length 17 miles shows that any solution to the travelling salesman problem for *all five* towns must have total length *at least* 36 miles.

Solutions and Comments

Section 1

1.1

(a) $2^6 \times 2^{-3} = 2^{6-3} = 2^3 = 8$.

(b) $2^n \div 2^3 = 2^{n-3}$.

(c) $k^4 \times k^7 \times (k^2)^{-1} = k^4 \times k^7 \times k^{-2} = k^{4+7-2} = k^9$.

1.2

(a) This is equivalent to

$$2x + 4 \leq 5x + 1$$

which simplifies to

$$3 \leq 3x, \text{ or } x \geq 1.$$

(b) This is equivalent to

$$\frac{1}{x} < \frac{1}{4}.$$

Assuming that x is positive, this simplifies to

$$4 < x, \text{ or } x > 4.$$

(The simplification step involves multiplying each side by $4x$, and this operation preserves the inequality sign only if $4x$ is positive. This is why we asked you to assume x positive.)

1.3

(a) $\mathbf{AB} = \begin{bmatrix} 11 & -4 & 2 \\ 25 & -14 & 10 \end{bmatrix}$.

(b) In a matrix product, the rows of the first matrix are combined with the columns of the second to produce the entries of the product. Thus the number of *columns* of the first matrix (which is the number of entries in each *row*) must equal the number of *rows* of the second matrix (which is the number of entries in each *column*). But **B** has three columns while **A** has two rows.

1.4 Using arithmetic modulo 2, we have:

(a) $$\mathbf{CC}_1 = 1 \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+0+1+0+1 \\ 1+0+1+0+0 \\ 1+0+1+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) $$\mathbf{CC}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+1+0+0 \\ 1+0+1+0+0 \\ 1+0+1+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

1.5

(a) $(0, 1, 0)$, $(1, 0, 0)$, $(1, 0, 1)$, $(1, 1, 1)$.

(b) The cube has twelve edges.

1.6 The equation $bk = vr$, along with the values of b, v and r gives $r = 3$. Thus,

$$\lambda = \frac{r(k-1)}{v-1} = \frac{3 \times 2}{6} = 1.$$

1.7 $u_3 = 4 + (1) = 5$;
$u_4 = 10 + (2 + 2) = 14$;
$u_5 = 28 + (5 + 4 + 5) = 42$;
$u_6 = 84 + (14 + 10 + 10 + 14) = 132$.

1.8 There are ten triangles: *abc*, *abd*, *abe*, *acd*, *ace*, *ade*, *bcd*, *bce*, *bde*, *cde*.

If there were n dots, there would be n ways of choosing the first corner, then $(n-1)$ ways of choosing the second and finally $(n-2)$ ways of choosing the third.

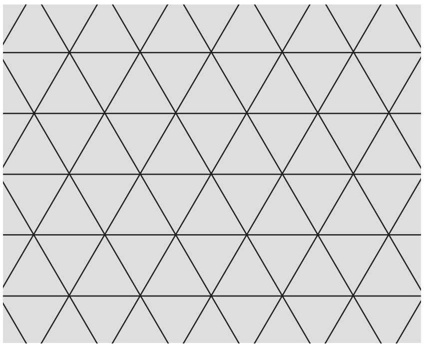
However, each of these $n(n-1)(n-2)$ sequences of choices describes a triangle with its corners given *in a particular order*, and the three corners of any triangle can be listed in six different orders. (For example, the first triangle of the ten listed above can be given as *abc*, *acb*, *bac*, *bca*, *cab* or *cba*.) Thus the answer is $\frac{n(n-1)(n-2)}{6}$. (If you have met binomial symbols before, you will recognize this as $\binom{n}{3}$.)

Section 2

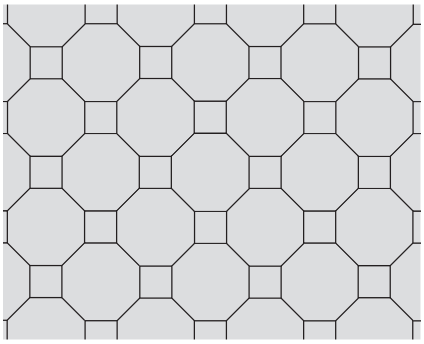
2.1 No, the map cannot be coloured with just three colours. Three colours are needed for the ring of five states surrounding Nevada (since if we try to colour them with two colours alternating, we find that two adjacent states in the ring must have the same colour). Now Nevada is adjacent to all of these states, and hence requires a fourth colour.

Although it may be difficult to see, actually the whole map can be coloured with just four colours.

2.2



(a)



(b)

(c) A regular pentagon has an angle of 180 degrees at each corner, and 360 degrees is not a multiple of 108 degrees. So it is not possible to fit other pentagons round any corner of any one pentagon.

2.3

(a) For network (1), the smallest number of links whose closure would separate the network is 2. The closure of any of the pairs: AB and AF; BC and FE; or CD and ED; would separate the network.

For network (2), the smallest number of links whose closure would separate the network is 3. AB, AE and AF; CB, CD and CE; DB, DC and DE; or FA, FB and FE; would separate the network.

(b) For network (1), the smallest number of exchanges whose closures would separate the remaining changes is 2. The closure of any of the pairs: B and E; B and F; C and E; or C and F; would separate the network.

For network (2), the smallest number of exchanges whose closure would separate the remaining exchanges is also 2. The only pair that would do this is: B and E.

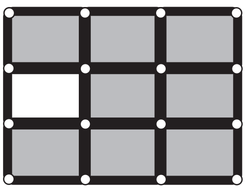
2.4

(a) We can, for example, send 7 units of fluid per unit time along the route $SADGT$ and 6 along the route $SCEHT$.

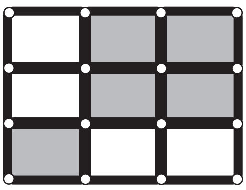
(b) We can, for example, augment the flow described above, by sending a further 2 units of fluid per unit time along the route $SBDGIT$.

(c) Each route from S to T passes through one of the pipelines DG, FG, HI and HT. Therefore, if we imagine drawing a line across these four pipelines, no more than $9 + 4 + 4 + 6 = 23$ units of fluid can cross that line per unit time.

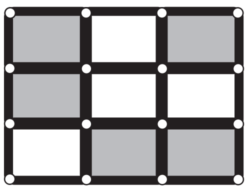
2.5



(a)



(b)



(c)

(a) One possibility is to start by removing the braces from the top-right and bottom-right rectangles. The remaining braces are clearly enough to keep the structure rigid. In fact, you can now also remove any *one* further brace as long as it is not either of the two in the middle row.

(b) The distortion shown on page 7 is the only possible distortion, and it distorts all four of the unbraced rectangles by the same angle. Thus, *any one* of these positions has the property that a brace in that position is enough to achieve rigidity.

(c) This framework is rigid, but the removal of any one of the five braces will destroy the rigidity.

2.6

(a) The other solutions are:

1: carpenter;	2: bricklayer;	3: plumber;	4: electrician
1: carpenter;	2: bricklayer;	3: decorator;	4: electrician
1: decorator;	2: bricklayer;	3: electrician;	4: carpenter
1: decorator;	2: bricklayer;	3: plumber;	4: carpenter
1: decorator;	2: bricklayer;	3: plumber;	4: electrician
1: decorator;	2: carpenter;	3: plumber;	4: electrician
1: decorator;	2: plumber;	3: electrician;	4: carpenter.

(b) Yes, it is still possible. All but the last one of the above solutions are still valid.

2.7

(a) The minimum connector that links the towns B, C, D and E comprises the links BD, CE and DE, of total length 27 miles.

(b) The minimum connector that links all five towns comprises the links AE, ED, AC and AB, of total length 26 miles. (Yes, linking all five towns can be done in a shorter length than linking just the four towns B, C, D and E!)

2.8 Any solution involves a round trip, and so can start and finish at any town we choose. Let us decide to start and finish it at B. Thus, the salesman must proceed from B to one of the other towns; *then* must cover the other four towns; *then* must return to B along a *different* road than that from which he left B. The first and last of these stages must be at least as long as the sum of the *shortest two* routes from $B(9 + 10 = 19$ miles), while the middle stage must be *at least as long* as a minimum connector for the towns other than B; which we have seen to comprise 17 miles. Thus, any solution for the five towns must have total length at least $19 + 17 = 36$ miles.